1. Multiple Choice A One correct answer each.

(a) State the order and the linearity or type of nonlinearity for the following PDE: $(u + u_{xx} + u_{yy})^{19} = u_y + 2022u$.

- ◯ First order, linear.
- ◯ First order, nonlinear. It is not quasilinear.
- ◯ Second order, nonlinear. It is quasilinear.
- ◯ Second order, nonlinear. It is not quasilinear.

(b) State the order and the linearity or type of nonlinearity for the following PDE: $(x^2 + 1)\Big(u_{xxx} + \sinh(3y)u_x\Big) = \ln(x^2 + 1)u$.

- ◯ Second order, nonlinear. It is quasilinear.
- ◯ Third order, nonlinear. It is quasilinear.
- ◯ Third order, linear.
- ◯ Second order, linear.

(c) The PDE $e^{x+y}u_{xx} + 2e^x u_{xy} - (e^{y-x} - 2)u_{yy} + 7u = \cos(x^2)$, is

- ◯ nowhere hyperbolic.
- ◯ parabolic when $x \neq y$.
- ◯ parabolic when $x = y$.
- ◯ everywhere elliptic.

(d) For which function $f : \mathbb{R} \to \mathbb{R}$ is the following second order linear PDE

$$\frac{1}{4}f(x)u_{xx} + u_{xy} + f(x)u_{yy} - f^2(x)yu_y = u$$

everywhere elliptic?

- ◯ $f(x) = \sin(x)$.
- ◯ $f(x) = \sin(x) + 3$.
- ◯ $f(x) = \frac{1}{2}\sin(x)$.
- ◯ $f(x) = 1$.

2. Multiple Choice B One correct answer each.

(a) By the *transversality condition*, the PDE

$$\begin{cases} (u-y)u_x + (x-y)u_y = \ln(u^2+1), & (x,y) \in \mathbb{R}^2, \\ u(0,y) = y^2, & y \in \mathbb{R}, \end{cases}$$

admits

- ◯ a global solution.
- ◯ a local solution for $(x,y) \neq (0,-1)$.
- ◯ a local solution for $(x,y) \notin \{(0,0),(0,1)\}$.
- ◯ no solutions at all.

(b) Let u be the weak solution of

$$\begin{cases} u_y + (\alpha u - u^2)u_x = 0, & \text{in } (x,y) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = 1, & \text{if } x < 0, \\ u(x,0) = 0, & \text{if } x \geq 0, \end{cases}$$

obtained by imposing the Rankine-Hugoniot condition, where $\alpha \in \mathbb{R}$ is some given constant. Then, u satisfies the entropy condition if

- ◯ $\alpha > \frac{4}{3}$.
- ◯ $\alpha < \frac{4}{3}$.
- ◯ $\alpha > -\frac{2}{3}$.
- ◯ $\alpha < \frac{2}{3}$.

(c) Denote with $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ the (open) unit disk. For which value of c is the PDE

$$\begin{cases} \Delta u = y^2 + c, & \text{in } D, \\ \partial_\nu u = x, & \text{on } \partial D, \end{cases}$$

solvable?

- ◯ $c = \frac{1}{2}$.
- ◯ $c = 0$.
- ◯ $c = -\frac{1}{4}$.
- ◯ $c = -\frac{\pi}{8}$.

3. Boxes Fill the white boxes with the final answer. Do not include your computations.

(a) Let $\alpha \in \mathbb{R}$ be a constant. Then, the function

$$u(x,y) = \begin{cases} 1, & \text{if } x \geq \alpha y, \\ 0, & \text{if } x < \alpha y, \end{cases}$$

is a weak solution of the conservation law

$$\begin{cases} u_y + (2u + u^3)u_x = 0, & (x,y) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = 1, & x \geq 0, \\ u(x,0) = 0, & x < 0, \end{cases}$$

if α is equal to

(b) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{2x+1}{x^2+y^2+1} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \left\{(x,y) \in \mathbb{R}^2 : 1 < \sqrt{x^2+y^2} < 2\right\}$. The maximum of u in D is equal to

(c) Let $u : [0,1] \times [0,+\infty) \to \mathbb{R}$ be a solution of the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } (0,1) \times (0,+\infty), \\ u(0,t) = u(1,t) = e^{-t}\sin(t), & \text{for } t \geq 0, \\ u(x,0) = \sin(\pi x), & \text{for } x \in [0,1]. \end{cases}$$

What is the maximum of u?

(d) Consider the one dimensional, homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in \mathbb{R}^2, \\ u(x,0) = g(x), & x \in \mathbb{R}, \\ u_t(x,0) = 1, & x \in \mathbb{R}, \end{cases}$$

where $g(x) = \ln(x^2+1)$ if $|x| < 9$ and $g(x) = 0$ otherwise. Where are the discontinuities of u?

4. Consider the following PDE in the unit square

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in [0,1] \times [0,1], \\ u_y(x,0) = \alpha, & x \in [0,1], \\ u_y(x,1) = 0, & x \in [0,1], \\ u_x(0,y) = 0, & y \in [0,1], \\ u_x(1,y) = \alpha^2, & y \in [0,1], \end{cases}$$

where $\alpha \in \mathbb{R}$.

(a) Find all values of $\alpha \in \mathbb{R}$ for which the above PDE is solvable.

(b) How many distinct solutions does the above PDE admit? Justify your answer.

(c) Find *all* solutions of the above problem.

If you were not able to solve point (a), set $\alpha = 1$.

(d) Show that the following system

$$\begin{cases} \Delta u(x,y) = 1, & (x,u) \in [0,1] \times [0,1], \\ u_y(x,0) = 1, & x \in [0,1], \\ u_y(x,1) = 0, & x \in [0,1], \\ u_x(0,y) = 0, & y \in [0,1], \\ u_x(1,y) = 1, & y \in [0,1], \end{cases}$$

admits no solution $u \in C^2([0,1] \times [0,1])$.

5. Consider the PDE

$$\begin{cases} u_y + (u^3+1)u_x = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

where $h(x) = 1$ if $x \le 0$, $h(x) = (1-x)^{\frac{1}{3}}$ if $0 < x < 1$ and $h(x) = 0$ if $x \ge 1$.

(a) Identify the flux function.

(b) Check the transversality condition: what can we say about the existence of classical solutions? (is it global/local, unique/not unique?).

(c) Compute the critical time of existence y_c.

(d) Solve the above conservation law using the Method of Characteristics.

(e) Find a weak solution everywhere defined.

6. Consider the disk $D := \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\right\}$. Let $u = u(x, y)$ be a function twice differentiable in D and continuous in $\bar{D}$, solving

$$\begin{cases} \Delta u(x, y) = 0, & \text{in } D, \\ u(x, y) = g(x, y), & \text{on } \partial D, \end{cases}$$

for some given function g.

(a) Suppose $g(x, y) = x^2 + y^2 + xy$.

i. Compute $u(0, 0)$.

ii. Compute $\max_{(x,y)\in\bar{D}} u(x, y)$ and $\min_{(x,y)\in\bar{D}} u(x, y)$.

Hint: recall the trigonometric identity: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

(b) Suppose now that g is any smooth function such that $g(x, y) \leq x^2 - y^2 + y$.

i. Show that $u(\frac{1}{2}, 0) \leq \frac{1}{4}$.

ii. Show that if $u(\frac{1}{2}, 0) = \frac{1}{4}$, then $g(x, y) = x^2 - y^2 + y$.

Hint: the function $x^2 - y^2 + y$ *is harmonic.*